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A Characterization of the Unicity Property in Best Approximation

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In this paper we consider the problem of characterizing those situations under which the best uniform linear approximation to an arbitrary continuous function is unique. The problem has been solved by Haar where the set of approximants is a finite dimensional subspace, but in this paper we generalize this by allowing the set of approximants to be any subset of a finite dimensional space. Some previous work has been done on this problem by Rice [2, p. 87 ff.] for a number of partial results.

I.

We consider a compact metric space $\bar{\mathbf{X}}$ and the space $C(\bar{\mathbf{X}})$ of real valued continuous functions on $\bar{\mathbf{X}}$. For $g \in C(\bar{\mathbf{X}})$ we define $\| g \| = \max_{x \in \bar{\mathbf{X}}} | g(x)|$. Functions in $C(\bar{\mathbf{X}})$ are to be approximated by linear combinations of n given linearly independent continuous functions $f_1(x), f_2(x),..., f_n(x)$. A linear combination of the $f_i(x)$ is represented by $a \cdot F(x)$ where $a \in R^n$, and $F(x) = [f_1(x),..., f_n(x)]$. Thus, $a \cdot F(x)$ is the usual dot product. Also if $a \in R^n$, $| a |$ denotes $(a \cdot a)^{1/2}$.

Let $B(g, K) = [a \in R^n: \| a \cdot F - g \| \leqslant K]$, denote the ball of center g and radius K intersected with the linear subspace of $f_i(x)$, and let P be a subset of R^n.

DEFINITION 1.1. A set $G \subset P$ is called a set of good approximations relative to P if there exists $g \in C(\bar{\mathbf{X}})$ and $K \geqslant 0$ such that $G = P \cap B(g, K)$. Further if g and K are such that $K = \inf_{a \in P} \| a \cdot F - g \|$, then G is said to be a set of best approximations relative to P.

We wish to emphasize that when we simply use the term "good (best) approximations," we mean relative to R^n (i.e., good (best) from the entire linear subspace F generated by the ordered basis $f_1(x),..., f_n(x)$).

DEFINITION 1.2. We say that P possesses the unicity property (with respect to F) if each set of best approximations relative to P consists of a single point.

In this paper we characterize those sets P which do not possess this property. The two situations in which P may fail to possess the unicity property are represented in Fig. 1 and 2.

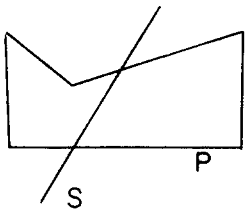

FIGURE 1

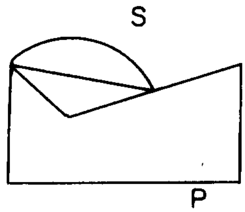

FIGURE 2

In Fig. 1 a set of best approximations, S, intersects P in more than one point. In Fig. 2 an n dimensional set of good approximations, S, intersects P in two or more points all of which lie on the boundary of S. Our characterization is based upon a characterization of these sets of best and good approximations. It is geometric in nature and shows that in general the set of directions attained by the vector $F(x) = [f_1(x), f_2(x),..., f_n(x)]$ plays an important role in determining whether or not P has the unicity property.

As an application we give an example to show that P may have the unicity property even though its complement contains a convex set whose boundary touches P in more than one point. This example is of interest in light of a conjecture in [2, p. 90].

II

As in [1] we let $(\text{B.A.})_g$ denote $B(g, N_c^*(g))$, the set of best approximations to g. Thus, $N_c^*(g)$ (c for Chebychev) equals $\inf_{a\in R^n} \| a \cdot F - g \|$.

THEOREM 2.1. *A necessary and sufficient condition that P not possess the unicity property is that either* (i) *there exists a set of best approximations containing more than one point of P or* (ii) *there exists a set of good approximations, with nonempty interior, which is such that P intersects its boundary in at least two points while not intersecting its interior.*

Proof. Sufficiency: If S is a set of best approximations containing more than one point of P, then $P \cap S$ is a set of best approximations relative to P, and, hence, P does not have the unicity property.

Let $S = B(g, K)$ be a set of good approximations with the properties (ii). If $b \in P$ and $\| b \cdot F - g \| < K$, b would have to be on the boundary of S. But then there would exist $a \notin S$ such that $\| a \cdot F - g \| < K$. Since this is impossible, $\| b \cdot F - g \| \geqslant K$. It follows that $S \cap P$ is a set of best approximations relative to P, and, thus, P does not have the unicity property.

Necessity: Let $G \subset P$ be a set of best approximations relative to P consisting of more than one point. For some g, $G = P \cap B(g, K)$ where $K = \inf_{a \in P} \| a \cdot F - g \|$. Let $S = B(g, K)$. If S is not a set of best approximations then $K > N_c^*(g)$ and, hence, the interior of S is nonempty. Further, if b were contained in $S \cap P$ and yet not on the boundary of S, we could choose $d \in (\text{B.A.})_g \subset S$ and a on the boundary of S such that $b = \lambda d + ua$, $\lambda > 0$, $u \geqslant 0$ and $\lambda + u = 1$. We would then have

$$\| b \cdot F - g \| \leqslant \lambda \| d \cdot F - g \| + u \| a \cdot F - g \| = \lambda N_c^*(g) + uK < K.$$

But, since this is not possible, $S \cap P$ is contained in the boundary of S.

To obtain a more useful characterization we must characterize the sets of good and of best approximations. In the "best" case this has been done in [1], and for the "good" case the analogous results are now developed.

If S is a compact convex set in R^n we let

$$N_c^m(S) = \sup_{x \in \bar{\mathbf{X}}}[\tfrac{1}{2}(\max_{a \in S} a \cdot F(x) - \min_{a \in S} a \cdot F(x))].$$

In [1] it is shown that if S is a set of best approximations,

$$N_c^m(S) = \min_g[N_c^*(g) \colon (\text{B.A.})_g = S].$$

The set of points at which this extreme value is attained is

$$[x \in \bar{\mathbf{X}} \colon \tfrac{1}{2}(\max_{a \in S} a \cdot F(x) - \min_{a \in S} a \cdot F(x)) = N_c^m(S)].$$

We denote this set by $E_F(S)$. Note that $N_c^m(S)$ and $E_F(S)$ are invariant under translation of S.

LEMMA 2.2. *If $S = B(g, K)$, then $K \geqslant N_c^m(S)$.*

Proof. The proof is the same as that of Lemma 1.4 in [1] except that $N_c^*(g)$ is replaced by K.

LEMMA 2.3. *If $S = B(g, K)$ there is a function $g_m \in C(\bar{\mathbf{X}})$ such that $S = B(g_m, N_c^m)$.*

Proof. In the proof of Theorem 1.6 of [1] we may replace $N_c^*(g)$ by K and omit the last paragraph and establish this result.

LEMMA 2.4. *$S = B(g, K)$ if and only if $S + a = B(g + a \cdot F, K)$.*

Proof. See [1] Lemma 2.1.

At this point the reader should be aware of what we mean by "enfold." A precise definition appears in [1]. Roughly, however, a set of real n dimensional vectors T enfolds a smooth compact n dimensional convex set $S \subset R^n$ containing the origin, if when the vectors in T are normalized to length 1 (call this set the normalization of T), there is for each tangent plane to S an inward unit normal contained in the closure of the normalization of T. When S is not smooth support hyperplanes replace the tangent planes. Further, if the convex set is not n dimensional T enfolds S if both the above criterion is satisfied by the projections of the vectors T on $L(S)$, the smallest linear subspace containing S, and, additionally, there is a sufficient supply of vectors orthogonal to $L(S)$ which are contained in the closure of the normalization of T.

Let S be a compact convex set in R^n with 0 as an interior point relative to $L(S)$, and let $K \geqslant N_c^m(S)$. We have the following theorem.

THEOREM 2.5. *A necessary and sufficient condition for the existence of $g \in C(\bar{\mathbf{X}})$ such that $S = B(g, K)$ is that there exist two closed subsets Q_1, $Q_2 \subset \bar{\mathbf{X}}$ with the following properties:*

(i) $Q_1 \cap Q_2 = \varnothing$ *if* $K > N_c^m(S)$,
$Q_1 \cap Q_2 = E_F(S)$ *if* $K = N_c^m(S)$.

(ii) $F(Q_1) \cup (-F(Q_2))$ *enfolds* S.

Proof. Sufficiency: The proof, with one exception, is the same as the proof of the sufficiency portion of Theorem 3.1 in [1]. Because we do not assume there exists an $x_0 \in Q_1 \cup Q_2$ such that $F(x_0) \in L(S)^\perp$, our construction of $g(x)$ does not allow us to say, that for all $a \in S$, $\| a \cdot F - g \| = K$.

Necessity: The proof is completed the same way as the necessity portion of Theorem 3.1 in [1], except that we do not assume $K = N_c^*(g)$, and we omit the paragraph containing the reference to Lemma 3.2.

To avoid the assumption that 0 is an interior point of S relative to $L(S)$, we use the following convention: If S is a compact convex set in R^n, choose $a \in S$ so that 0 is an interior point of $S - a$ relative to $L(S - a)$, and then say that T enfolds S if T enfolds $S - a$. Also, we define the dimension of S, $\dim S$, to be the dimension of $L(S - a)$.

It is easy to show that if Q_1 and Q_2 are closed, $\dim S < n$ and $F(Q_1) \cup (-F(Q_2))$ enfolds S, then there exists an $x_0 \in Q_1 \cup Q_2$ such that $F(x_0) \in L(S)^\perp$. Now Lemma 2.3, which has its analog in [1], implies that S is a set of good approximations if and only if for some $g \in C(\bar{\mathbf{X}})$ we have

$S = G(g, N_c^m(S))$. We see from Theorem 2.5, Lemma 2.4, and the corresponding results in [1], that for any compact convex set S the following two theorems are valid.

THEOREM 2.6. *A necessary and sufficient condition for S to be a set of good approximations is that there exist two closed sets Q_1, $Q_2 \subset \bar{\mathbf{X}}$ with the following properties*:

(i) $Q_1 \cap Q_2 = E_F(S)$.

(ii) $F(Q_1) \cup (-F(Q_2))$ *enfolds* S.

THEOREM 2.7. *A necessary and sufficient condition for S to be a set of best approximations is that there exist two closed sets Q_1, $Q_2 \subset \bar{\mathbf{X}}$ with the following properties*:

(i) $Q_1 \cap Q_2 = E_F(S)$.

(ii) $F(Q_1) \cup (-F(Q_2))$ *enfolds* S.

(iii) *If* $\dim S = n$ *then* $F(x)$ *vanishes at some point of* $Q_1 \cup Q_2$.

Our characterization is now an immediate consequence of Theorems 2.1, 2.6, and 2.7.

THEOREM 2.8. *A necessary and sufficient condition that P not possess the unicity property is that there exist* (a) *a compact convex set $S \subset R^n$ which contains at least two points of P, and* (b) *two closed subsets Q_1, $Q_2 \subset \bar{\mathbf{X}}$ such that the following conditions hold*:

(i) $Q_1 \cap Q_2 = E_F(S)$.

(ii) $F(Q_1) \cup (-F(Q_2))$ *enfolds* S.

(iii) *If* $\dim S = n$ *and P intersects the interior of S, then $F(x)$ vanishes at some point of* $Q_1 \cup Q_2$.

If $\{f_i(x)\}$ form a Chebychev set then each set of best approximations consists of a single point. Thus, from Theorems 2.1 and 2.6 we have Theorem 2.9.

THEOREM 2.9. *If $\{f_i(x)\}$ form a Chebychev set then a necessary and sufficient condition that P not possess the unicity property is that there exist* (a) *an n dimensional compact convex set $S \subset R^n$ whose boundary intersects P in at least two points and whose interior lies in the complement of P and* (b) *two closed subsets Q_1, $Q_2 \subset \bar{\mathbf{X}}$ such that the following conditions hold*:

(i) $Q_1 \cap Q_2 = E_F(S)$.

(ii) $F(Q_1) \cup (-F(Q_2))$ *enfolds* S.

To illustrate this theorem we consider the following example.

EXAMPLE 2.9. Let $n = 2$, $\bar{\mathbf{X}} = [0, \frac{1}{2}]$ and $F(x) = [1, x]$. $a = (a_1, a_2)$ and $P = [a: a_2 \leqslant | a_1 |]$.

Let S be a two dimensional compact convex set whose boundary touches the boundary of P in at least the two points b and d, and whose interior does not intersect P.

The boundary of P is made up of two line segments so that if b, d lie on the same one of these segments the boundary of S must contain a subsegment of this segment. But since $F(\bar{\mathbf{X}})$ does not contain a vector perpendicular to this segment it is not possible to find $Q_1, Q_2 \subset \bar{\mathbf{X}}$ such that $F(Q_1) \cup (-F(Q_2))$ enfolds S.

If b, d lie on different segments, let b^* be on the a_2 axis on or below the segment connecting b and d and on the boundary of S.

A support hyperplane to S through b^* is a line through b^* which divides R^2 into two half planes in such a way that b and d are not in opposite half planes. But since such a line would have to have a slope between -1 and 1 it is again not possible (since $F(\bar{\mathbf{X}})$ contains no vector perpendicular to such a line) to find $Q_1, Q_2 \subset \bar{\mathbf{X}}$ such that $F(Q_1) \cup (-F(Q_2))$ enfolds S. We conclude, by Theorem 2.9, that P possesses the unicity property.

REFERENCES

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